

The Distribution of Wealth with Imperfect Altruism*

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In this paper, we study the distribution of wealth in an economy with infinitely lived families. Individual generations of each family may or may not be altruistic. This is represented as a preference shock which follows a first-order Markov process within each family, a feature representing *imperfect altruism*. Altruistic individuals care about the welfare of their children and are likely to leave bequests; selfish ones do not.

This results in a non-trivial distribution of wealth among families at any point in time. We study an economy with a risk-free, linear production technology and show that a stationary distribution of wealth exists. This distribution is discrete and approximates the pareto distribution under additional restrictions. We also characterize conditions on the production technology which yields perpetual growth with increasing inequality. *Journal of Economic Literature* Classification Numbers: D31, D91, E21. © 1998 Academic Press

1. INTRODUCTION

In this paper, we study the distribution of wealth in an economy where individuals are altruistic; importantly, this altruism is imperfect.

Altruistic individuals care about the welfare of their children, and are likely to leave bequests. This is precisely what makes for a non-trivial distribution of wealth among families at any point in time, as some receive larger inheritances than others. We take the view that such altruism is not

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a universal phenomenon; some individuals are inherently selfish, and would choose to consume their entire wealth and earnings. Even in the wealthiest families, the appearance of a single profligate generation is enough to draw wealth down to zero. We choose to model this as a preference shock, where each generation of infinitely-lived dynasties has some probability of being selfish. To keep things simple, we assume that the preference shocks follow a first order Markov process; so that the proportion of altruists in the population is constant over time, at the corresponding stationary distribution. Production possibilities are also assumed to be constant and riskless.

We then study the evolution of the wealth, and its economy-wide distribution. A stationary distribution of wealth exists. Our main interest here is in the characteristics of the wealth distribution, which can be analyzed in fairly general conditions. This distribution has the property of declining frequency—that larger holdings of wealth occur with lower frequency, and the frequencies are closely approximated by that of the Pareto distribution. Starting from any initial wealth holdings, the actual distribution converges to the stationary one over time. A non-degenerate stationary distribution implies, of course, that some degree of wealth inequality is preserved in the long run. The stationary distribution is degenerate—implying perfect equality in the long run—if and only if capital carries a low enough rate of return. The aggregate distribution is constant, even though wealth levels of each component family fluctuate. If the rate of return on capital is sufficiently high, the economy experiences perpetual growth in per capita output, while the distribution of wealth approaches the stationary distribution. Importantly, the wealth levels of individual families fluctuate substantially, even though the aggregate distribution is constant.

The paper is organized as follows. We set out the model in Section 2, and analyze the bequest decision in the presence of imperfect altruism in Section 3. Section 4 characterizes the stationary distribution of wealth. We investigate the issue of whether this distribution is compatible with convergence to a stationary outcome in Section 5, showing that failure of such convergence is compatible with a stationary wealth distribution and perpetual growth. Section 6 concludes.

2. THE ECONOMY

The economy has an infinite horizon, while individuals are finitely lived. Time is discrete, with $t = \dots, 1, 2, \dots$. There are a continuum of infinitely lived families, indexed by $h \in (0, 1]$. For simplicity, there is exactly one member of each family born in each period, t , so that there is no population growth.

The index h, t refers to an individual belonging to family h , born at time t . Each such individual lives for one period, and is endowed with 1 unit of labor. The aggregate supply of labor in the economy is 1 at each t .

Individuals receive bequests from their parents; and spend their total wealth, which consists of inheritance plus labor income, on own consumption and on bequests to their children. All bequests are in the form of capital, which earns interest. Let x_t^h be the bequest left by individual (h, t) , and w_t her wage income. The total wealth of individual (h, t) is $y_t^h = w_t + (1 + r_t) x_{t-1}^h$, where savings earn interest at rate r_t . The budget constraint faced by (h, t) is

$$c_t^h + x_t^h \leq y_t^h = w_t + (1 + r_t) x_{t-1}^h.$$

Bequests are non-negative, with $x_t^h \geq 0$ for all (h, t) . The amount x_t^h depends on the wealth as well as the preferences of (h, t) , which we describe next.

2.1. Imperfect Altruism

An individual is altruistic if she values the welfare of her offspring. We take the view that altruism is not universal; in particular, that some proportion of the population are altruistic, while the remainder are selfish, and care only about their own welfare. They rationally choose to leave zero bequests. Altruism, or selfishness, may be a persistent characteristic of families, even though every family faces the prospect that some generation may turn out to be selfish. We model this as a preference shock within a family which is assumed to follow a Markov process. Importantly, altruists need to choose their bequest levels irrespective of the preferences of their offspring.

We model this as uncertainty with respect to future preferences. Specifically, let $s(h, t)$ be the state of preferences of (h, t) , where $s(h, t) \in \{0, 1\}$. The state $s = 1$ corresponds to altruism and state 0 to selfishness. Individuals value consumption by the instantaneous utility function $U(c)$.

The realization of $s(h, t)$ determines the rate at which the utility of the next generation is discounted. Specifically, $\beta_t^h = s(h, t) \beta$ where $\beta > 0$ is the discount factor of all altruistic individuals. We assume that $s(h, t)$ follows a Markov process, with

$$\pi_{s's} = \text{Prob}(s(h, t+1) = s' \mid s(h, t) = s), \quad s, s' \in \{0, 1\}$$

for every $h \in (0, 1]$. By definition, π_{11} is the probability that an altruistic generation is followed by another, and $\pi_{01} \equiv 1 - \pi_{11}$ is the probability that an altruistic generation is followed by a selfish one; similarly for $\pi_{00}, \pi_{10} \equiv 1 - \pi_{00}$.

The stochastic process has a stationary distribution; the proportion of altruists in the population at the stationary distribution is π , which must satisfy $\pi\pi_{11} + (1 - \pi)\pi_{10} = \pi$. It follows that

$$\pi = \frac{1 - \pi_{00}}{2 - \pi_{00} - \pi_{11}}.$$

We assume that $0 < \pi_{00} < 1$ and $0 < \pi_{11} < 1$, which implies $0 < \pi < 1$.

If $\pi_{00} = \pi_{11} = 1$, the initial proportion π of altruistic families is constant, as in Muller and Woodford [21], and Michel and Pestieau [20]. The situation $\pi_{11} = 1$, $\pi_{00} < 1$ implies $\pi = 1$ at the stationary distribution, which corresponds to an economy with effectively infinitely-lived agents, e.g. Barro [6]. In that situation, the constraint that bequests be non-negative is important, as shown by Weil [25]. In discussing the relevance of fiscal policies in economies with altruistic agents, Abel and Bernheim [2] demonstrate that some form of imperfection is necessary, and isolate $\pi < 1$ as one possible representation of such imperfection.

2.2. Preferences and the Bequest Motive

Selfish individuals care only about their own welfare, so that their preferences are represented by $U(c_t^h)$, as usual. Altruists care about the utility of the next generation. This leads to a recursive representation of preferences, as we show next.

Let $C_t^h = (c_t^h(0), c_t^h(1)) \gg 0$ be a contingent consumption plan, where $c_t^h(s)$ is the planned consumption if $s(h, t) = s$. Define

$$\mathcal{C}_t^h = \{C_{t+i}^h; i = 0, 1, 2, \dots\}$$

as the forward consumption plan of family h at time t . We write $V_s(\mathcal{C}_t^h)$ as the value of this consumption plan to an individual of type s .

Selfish individuals value only their own consumption. The utility of consumption is $U(c)$, so that

$$V_0(\mathcal{C}_t^h) \equiv V(\mathcal{C}_t^h; s(h, t) = 0) = U(c_t^h(0)).$$

Altruists value the utility of their offspring; however, this utility may be $V_0(\mathcal{C}_{t+1}^h)$, if their offspring is selfish, or $V_1(\mathcal{C}_{t+1}^h)$, the utility of the remainder stream to an altruist. This leads to the recursive definition

$$\begin{aligned} V_1(\mathcal{C}_t^h) &\equiv V(\mathcal{C}_t^h; s(h, t) = 1) = U(c_t^h(1)) + \beta EV(\mathcal{C}_{t+1}^h) \\ &= U(c_t^h(1)) + \beta([\pi_{11} V_1(\mathcal{C}_{t+1}^h) + (1 - \pi_{11}) V_0(\mathcal{C}_{t+1}^h)]). \end{aligned}$$

Assumption 1. The utility function $U(c): c \in \mathbb{R}_+$ is increasing, bounded, strictly concave, and twice continuously differentiable, satisfying $\lim_{c \rightarrow 0} U'(c) = \infty$ and $\lim_{c \rightarrow \infty} U'(c) = 0$. Further, $0 < \beta\pi_{11} < 1$.

Remarks. • The value of a forward consumption plan $\mathcal{C}_t = \{(c_{t+i}(0), c_{t+i}(1)) \in \mathbb{R}_+^2 : i = 0, 1, 2, \dots\}$ to individual (h, t) is

$$V_{h,t}(\mathcal{C}_t) = s(h, t) V_1(\mathcal{C}_t) + (1 - s(h, t)) V_0(\mathcal{C}_t); \quad s(h, t) \in \{0, 1\}$$

where

$$V_0(\mathcal{C}_t) \equiv U(c_t(0))$$

and

$$V_1(\mathcal{C}_t^h) \equiv \sum_{i=0}^{\infty} (\beta\pi_{11})^i [U(c_{t+i}^h(1)) + \beta(1 - \pi_{11}) U(c_{t+i+1}^h(0))].$$

$V_1(\mathcal{C})$ is obtained by recursive substitution in the equation $V_1(\mathcal{C}_t) = U(c_t(1)) + \mathbf{E}V_S(\mathcal{C}_{t+1})$.

• The assumptions on U are strong, but standard (e.g. Stokey and Lucas [24], Chapter 4). They are clearly sufficient to ensure that forward consumption plans have finite value to altruists. Note that the modified discount rate is $\beta\pi_{11}$ in V_1 , which accounts for the probability of the events that future generations continue to be altruistic.

• A consumption plan \mathcal{C} is *stationary* if $c_t(s) = c(s)$ for all t and $s \in \{0, 1\}$. The value of a stationary consumption plan \mathcal{C} is

$$V_0(\mathcal{C}) = U(c_0)$$

and

$$V_1(\mathcal{C}) = \frac{U(c_1) + \beta(1 - \pi_{11}) U(c_0)}{1 - \beta\pi_{11}}.$$

• To maintain comparability with standard representations, note that the valuation of a state-independent consumption plan $\bar{\mathcal{C}}_t = \{c_{t+i}(s) = \bar{c}_{t+i} : i = 0, 1, 2, \dots\}$ is $V_1(\bar{\mathcal{C}}_t) = U(\bar{c}_t) + \sum_{i=1}^{\infty} \beta^i \pi_{11}^{i-1} U(\bar{c}_{t+i}^h)$.

2.3. Technology, Prices, and Incomes

We assume that the production technology is of a particularly simple form:

$$Y_t = \rho K_t + wL_t; \quad \rho > 0; \quad w > 0.$$

We assume, further, that production is carried out by competitive firms, which maximize profits. It follows that the real wage rate $w_t = w$ at each t . The aggregate supply of labor is 1 in each period. Labor market clearing implies $L_t = 1$.

Capital depreciates at rate δ each period, so that

$$K_{t+1} = (1 - \delta) K_t + I_t,$$

with I_t being the level of investment each period.

Firms choose capital to maximize operating profits plus the resale value of the firm. It follows that

$$r_t = \rho - \delta.$$

We assume that $0 < \delta \leq 1$, implying $\rho > r \geq \rho - 1$.

2.4. *Equilibrium and Dynamics*

The intertemporal behavior of aggregate output in this economy is driven by the accumulation of capital. In this economy, savings coincide with bequests.

At time t , household h , t chooses their bequest level x_t^h , subject to

$$x_t^h + c_t^h = w + (1 + r) x_{t-1}^h.$$

The quantity x_t^h is affected by preferences $s(h, t)$ as well as x_{t-1}^h , which determines initial wealth. Aggregate savings at time t are $X_t = \int_h x_t^h dh$.

Households hold their savings in the form of equity in firms. The value of a firm is just the replacement value of its capital stock. Capital market clearing implies that X_t equal investment plus depreciated capital stock:

$$X_t(1 - \delta) K_t + I_t = K_{t+1}.$$

Define F_t to be the distribution of wealth at t : $F_t(x) = \int_{x_t^h \leq x} dh$ for each $x \geq 0$. By definition, F_t is a probability distribution on \mathbb{R}_+ , with $X_t = \int_0^\infty x dF_t$. The evolution of capital stocks is described by

$$K_{t+1} = X_t = \int x dF_t.$$

Clearly, the dynamic behavior of output and capital stocks is determined by the evolution of the wealth distribution F_t . Recall that x_t^h is determined by x_{t-1}^h and $s(h, t)$.

2.5. *Stationary Equilibria*

The dynamical properties of this economy depend on the evolution of capital stocks and of the aggregate distribution of wealth x_t^h . Equilibrium

paths in this economy are sequences $Y_t > 0$, $K_t \geq 0$ and distributions F_t , which satisfy the market clearing and capital accumulation conditions. Let F_0 be the initial distribution of wealth, inherited by generation 1. The evolution of F_1, F_2, \dots , depends on the initial distribution F_0 . We are particularly interested in stationary equilibria, which require that the aggregate levels of output, capital and the distribution of wealth be time-invariant. The actual wealth and consumption levels of each family can vary over time on such a path.

DEFINITION 1. A *stationary equilibrium* is a level of capital stock $K \geq 0$, of output $Y > 0$, and a distribution of wealth F on \mathbb{R}_+ such that

1. At each t , x_t^h is the optimal bequest chosen by (h, t) , given $s(h, t)$ and x_{t-1}^h , and facing stationary prices w and $r = \rho - \delta$.
2. $F_t(x) = \int_{x_t^h \leq x} dh = F(x)$ at each t and for each $x \in \mathbb{R}_+$.
3. $K = \int x dF$ and $Y = \rho K + w$.

Condition (1) says that each family chooses their optimal bequest level. Condition (2) is satisfied if the optimal choice of each h is compatible with the stationary distribution at t as well as $t-1$. Condition (3) is just the capital accumulation and goods market clearing conditions.

3. THE BEQUEST DECISION

An individual (h, t) receives bequest x_{t-1}^h , and supplies 1 unit of labor elastically, facing the wage rate w . This individual knows her own type, and chooses c_t^h, x_t^h to maximize utility subject to the budget constraint $c_t^h + x_t^h = w + (1+r)x_{t-1}^h$, where r is the stationary rate of interest.

This individual knows her own type, $s(h, t)$, and chooses c_t^h, x_t^h which is optimal for her type, given wealth $w + (1+r)x_{t-1}^h$. The utility of bequest x to the next generation depends on what their preferences are, which affects their own bequest choice, and so on. In choosing x_t^h optimally, individuals need to solve the full infinite horizon program \mathbf{P}_s :

$$(\mathbf{P}_s): \max_{\mathcal{C}_t^h} V_s(\mathcal{C}_t^h)$$

where the consumption plan \mathcal{C}_t^h satisfies the following constraints:

$$c_{t+i}^h(s) + x_{t+i}^h(s) = w + (1+r)x_{t+i-1}^h;$$

$$x_{t+i}^h(s) \geq 0.$$

Each such individual takes as given the initial bequest level, x_{t-1}^h as given as well as stationary prices w, r .

Clearly, the problem \mathbf{P}_0 is trivial, and yields the solution $x_t^h = 0$ irrespective of the levels of x_{t-1}^h and prices. The altruists' decision is the outcome of the dynamic program \mathbf{P}_1 , which takes into account that $x_{t+i}^h = 0$ whenever $s(h, t+i) = 0$.

We want to characterize the properties of bequest functions. Preferences are recursive, which allows the bequest function to be time-invariant. The optimization problem \mathbf{P}_1 facing an altruist at any time can be represented by a Bellman equation

$$v(x) = \max_{0 \leq y \leq w + (1+r)x} U(w + (1+r)x - y) + \beta(1 - \pi_{11}) U(w + (1+r)y) + \beta\pi_{11}v(y).$$

THEOREM 1. *Let $A = (-w/(1+r), \infty)$, $w > 0$ and $r > -1$; and suppose Assumption 1 holds. Then, the value function v defined by*

$$v(x) = \max_{0 \leq y \leq w + (1+r)x} U(w + (1+r)x - y) + \beta(1 - \pi_{11}) \times U(w + (1+r)y) + \beta\pi_{11}v(y)$$

exists for each $x \in A$. It is increasing, strictly concave, and continuously differentiable for $x \in A$.

Proof. The proof is standard. Let $F(x, y) = U(w + (1+r)x - y) + \beta(1 - \pi_{11}) U(w + (1+r)y)$. Note that A is a convex subset of \mathbb{R} ; for each $x \in A$, the interval $I(x) \equiv [0, w + (1+r)x]$ is non-empty, compact, convex, and depends continuously on x . The function F is bounded and continuously differentiable in the interior of $B = \{(x, y): y \in I(x), x \in A\}$. It follows from Theorem 4.8 of Stokey and Lucas [24] that v exists, is increasing and strictly concave, and from Theorem 4.11 that it is continuously differentiable in the interior of A . ■

Remarks. • The value function $v(x)$ is well defined on $A = (-w/(1+r), \infty)$, even though bequest are non-negative. We need this to ensure the differentiability of v at $x = 0$, which is an interior point of A .

- The period utility function is

$$W(c_t, x_t) = U(c_t) + \beta(1 - \pi_{11}) U(w + (1+r)x_t),$$

which is similar to that of a consumer who derives utility directly from her wealth level as well as consumption (e.g., Kurz [15])

• An alternative form of impure altruism studied by Andreoni [3] and Abel and Bernheim [2] assumes that individuals derive utility directly from the size of the bequest, so that the period utility function writes as $u(c_t) + v(x_t)$. We should note that the function W mimics this representation, though the form of v is not arbitrary. The formal equivalence of the two representations is noted in Abel and Bernheim (*op. cit.*).

3.1. The Bequest Function

We want to study the properties of stationary bequest functions $\phi(x)$, which represents the optimal bequest level of an altruist who inherits x . This is just the optimal policy function of the problem \mathbf{P}_1 . Note that the policy function $\phi(x)$ is defined for $x \geq 0$.

COROLLARY 1. *Let $\phi(x) = \{y \in [0, w(1+r)x]: v(x) = U(w + (1+r)x - y) + \beta(1 - \pi_{11})U(w + (1+r)y) + \beta\pi_{11}v(y)\}$ for each $x \in \mathbb{R}_+$, and suppose Assumption 1 holds. Then*

1. ϕ is continuous and single valued.
2. $\phi(x) \in [0, w + (1+r)x]$ for each $x \in \mathbb{R}_+$.
3. $v'(x) = (1+r)U'(w + (1+r)x - \phi(x))$ for each $x \geq 0$.

Proof. The Proof follows from Theorem 4.11 of Stokey and Lucas (*op. cit.*), (see also Mirman and Zilcha [19], Benveniste and Scheinkman [8]), and noting that $v'(x) = F_x(x, \phi(x))$ for $x \in A$. Note that $\mathbb{R}_+ \subset A$, ensuring differentiability of v at each $x \geq 0$ by Theorem 1. Finally, $U'(0) = \infty$ which ensures that $\phi(x) < w + (1+r)x$ at the optimal solution. ■

THEOREM 2. *The bequest function $\phi(x)$ has the following properties:*

1. $\phi(x)$ is monotone non-decreasing in x , i.e. $x > x' \Rightarrow \phi(x) \geq \phi(x')$, with strict inequality whenever $\phi(x') > 0$.
2. $\phi(x) = 0 \Rightarrow \phi(x') = 0$ for $0 \leq x' < x$.
3. $\phi(0) = 0$ if, and only if, $(1+r) \leq 1/\beta$.

Proof. Define

$$G(x, y) = -U'(w + (1+r)x - y) + \beta(1 - \pi_{11})(1+r) \\ \times U'(w + (1+r)y) + \beta\pi_{11}v'(y).$$

From the definitions, $G(x, \phi(x)) \leq 0$ for each $x \geq 0$. Further, $\phi(x) > 0$ only if $y = \phi(x)$ is an interior solution to the optimization problem, so that $\phi(x) > 0 \Rightarrow G(x, \phi(x)) = 0$. Note also that U is strictly concave by assumption, and v is strictly concave by Theorem 1. It follows that G is strictly decreasing in y , and strictly increasing in x .

If $x' \geq 0$ is such that $\phi(x') > 0$, we have that $G(x, \phi(x')) > G(x', \phi(x')) = 0$ for $x > x'$. $G(x, y)$ is continuous and decreasing in y , so that $G(x', y) \leq 0$ only if $y > \phi(x')$. It follows that $\phi(x) > \phi(x')$.

If $\phi(x) = 0$, we have a boundary solution, so that $G(x, 0) \leq 0$, and $G(x', 0) \leq G(x, 0) \leq 0$ for $0 \leq x' < x$. It follows that $\phi(x) = 0 \Rightarrow \phi(x') = 0$ for $0 \leq x' < x$.

From the previous argument, $\phi(0) = 0$ if and only if $G(0, 0) \leq 0$. We can write

$$G(0, 0) = -U'(w) + \beta(1 - \pi_{11})(1 + r) U'(w) + \beta\pi_{11}v'(0).$$

From Corollary 1, $v'(0) = (1 + r) U'(w)$ whenever $\phi(0) = 0$. Further, $u'(w) > 0$ for finite w so that

$$G(0, 0) \leq 0 \Rightarrow U'(w)(-1 + \beta(1 + r)) \leq 0 \Leftrightarrow \beta(1 + r) \leq 1.$$

Suppose now that $\phi(0) = y_0 > 0$. Then $g(0, y_0) = 0$, and

$$\begin{aligned} 0 &= G(0, y_0) \\ &= -U'(w - y_0) + \beta(1 - \pi_{11})(1 + r) U'(w + (1 + r) y_0) + \beta\pi_{11}v'(y_0) \\ &< -U'(w - y_0) + \beta(1 - \pi_{11})(1 + r) U'(w - y_0) + \beta\pi_{11}v'(0), \end{aligned}$$

as U' and v' are strictly decreasing functions. The proof is completed by noting that $v'(0) = (1 + r) U'(w - y_0)$ from Corollary 1, so that

$$G(0, y) = 0 \Rightarrow \beta(1 + r) > 1$$

whenever $y > 0$. ■

Remarks. • Theorem 1 and Corollary 1 are results standard from the theory of discounted dynamic programming. Theorem 2 characterizes the optimal policy of an altruist.

- We choose to represent the optimal policy by the bequest function ϕ , where an altruist who inherits x will choose to leave bequest $x' = \phi(x)$. The function has the following property: if $\phi(0) = 0$, it is equal to 0 for some range $[0, x_*]$, and positive thereafter. Further, in the range (x_*, ∞) , it is strictly increasing.

- Theorem 2 also offers a complete characterization of the property $\phi(0) = 0$. This is equivalent to the familiar condition that the subjective discount rate $(1 - \beta)/\beta$ be at least as great as the interest rate r . Note that this is a restriction on β , even though the modified discount rate for the dynamic program \mathbf{P}_1 is $\beta\pi_{11}$.

- If $r < (1 - \beta)/\beta$, individuals with low wealth are constrained by the non-negativity of bequests. They would like to borrow from the wealth of

future generations but are assumed unable to do so by legal restrictions. Wealthy individuals—those with $x^h > x_*$, will be willing and able to behave bequests.

The function ϕ , and the realization of s , determines the evolution of wealth as follows. Suppose that they have initial wealth x_0^h , inherited by the first generation. Then,

$$x_t^h = s(h, t) \phi(x_{t-1}^h); \quad s(h, t) \in \{0, 1\}.$$

The evolution of the aggregate distribution of wealth is determined by this equation, and the assumptions that $s(h, t)$ follows a Markov process for each h , independently of $\{x_{t-i}^h\}$ and that there is no aggregate risk, so that the proportion of families with $s(h, t+1) = s'$, $s(h, t) = s$ is exactly $\pi_{s's}$.

3.2. Fixed Points of the Bequest Function

We study the evolution of wealth, as determined by $\phi(x)$, the bequest function. The property that $\phi(x)$ possess a positive, finite fixed point is important in determining the properties of the stationary distribution of wealth.

DEFINITION 2 (Property (F)). The bequest function ϕ possesses property (F) if $\phi(\hat{x}) = \hat{x}$ for some $0 < \hat{x} < \infty$.

Recall that the condition $\beta(1+r) \leq 1$ is equivalent to $\phi(0) = 0$, so that $x = 0$ is a fixed point of F . If this condition fails, any fixed point of ϕ must be strictly positive. ϕ is continuous, and $\phi(0) > 0$, so that property (F) fails if $\phi(x) > x$ for each $x \geq 0$.

THEOREM 3. Let $\gamma(x) = (U'(w + (1+r)x))/(U'(w + rx))$ for each $x \geq 0$, and $\theta(r) = (1 - \beta\pi_{11}(1+r))/(\beta(1 - \pi_{11})(1+r))$. The bequest function $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ has property (F) if and only if

$$1 > \theta(r) \geq \inf_{x \geq 0} \gamma(x).$$

Proof. We prove the statement in a series of Lemmas.

LEMMA 1. If ϕ has property (F), every positive fixed point \hat{x} of ϕ satisfies

$$\theta(r) = \gamma(\hat{x}).$$

Proof. Let $G(x, y)$ be as defined in the proof of Theorem 2. Note that $G(\hat{x}, \hat{x}) = 0$ whenever $\hat{x} = \phi(\hat{x}) > 0$. Further, $v'(\hat{x}) = (1+r)U'(w + r\hat{x})$. The result obtains from substitution. ■

LEMMA 2. *The bequest function $\phi(x)$ has property (F) only if*

$$1 > \theta(r) \geq \inf_{x > 0} \gamma(x).$$

Proof. The proof follows from Lemma 1, noting that $\gamma(x) < 1$ for each $x > 0$ whenever U is strictly concave. ■

LEMMA 3. *Suppose $\beta(1+r) > 1$. Then, $\phi(x)$ does not possess property (F) only if*

$$\gamma(x) > \theta(r)$$

for each $x > 0$.

Proof. Define

$$\begin{aligned} H(x) &= \beta(1 - \pi_{11})(1+r) U'(w + (1+r)x) + \beta\pi_{11}v'(x) \\ &\equiv G(x, x) + U'(w + rx). \end{aligned}$$

The function H is strictly decreasing. Further,

$$H(\phi(x)) = \beta(1 - \pi_{11})(1+r) U'(w + (1+r)\phi(x)) + \beta\pi_{11}v'(\phi(x)).$$

The restriction $G(x, \phi(x)) = 0$ implies

$$H(x) = \beta(1 - \pi_{11})(1+r) U'(w + (1+r)x) + \beta\pi_{11}(1+r) H(\phi(x)).$$

If $\phi(0) > 0$, property (F) fails if and only if $H(\phi(x)) < H(x)$ for each $x \geq 0$, which implies

$$(1 - \beta\pi_{11}(1+r)) H(x) < \beta(1 - \pi_{11})(1+r) U'(w + (1+r)x).$$

This condition is true if $\beta(1+r)\pi_{11} \geq 1$, implying $\theta(r) \leq 0$. Suppose $\beta(1+r)\pi_{11} < 1$, in which case, this implies

$$\theta(r) H(x) < U'(w + (1+r)x).$$

The result follows from the observation that $H(x) = G(x, x) + U'(w + rx) < U'(w + rx)$ whenever $\phi(x) > x$. ■

This completes the proof. ■

COROLLARY 2 (Uniqueness). *If $\gamma(x)$ is strictly decreasing in x , the function ϕ has at most one fixed point. Further, property (F) is equivalent to*

$$1 > \theta(r) > \gamma(\infty).$$

Proof. From Lemma 1, every fixed point \hat{x} satisfies $\gamma(x) = \phi(r)$. This equation has at most one solution in x if γ is strictly monotone. If γ is strictly decreasing, $\hat{x} < \infty \Leftrightarrow \gamma(x) < \gamma(\inf) = \inf_x \gamma(x)$. ■

Remarks. • Lemma 2 establishes that property (F) implies $\phi(0) > 0$; that is, 0 is the unique fixed point of ϕ whenever $\beta(1+r) \leq 1$. This implies that ϕ either has a unique fixed point at 0, or it possesses property (F), or it is true that $\phi(x) > x$ for each $x \geq 0$.

- The inequality condition in Lemma 2 rewrites as

$$1 < \beta(1+r) \leq \frac{1}{\pi_{11} + (1 - \pi_{11}) \inf_x \gamma(x)} \leq \frac{1}{\pi_{11}}.$$

It follows that if $(1+r) > 1/\beta\pi_{11}$, ϕ cannot have property (F). The requirement that $\beta(1+r)$ be bounded above as well as below can also be violated if β is sufficiently large.

- Lemma 2 establishes that the bounds are necessary for (F) to hold. Lemma 3 establishes that they are sufficient. Recall that $\theta(r) < 1 \Leftrightarrow \phi(0) > 0$.

- From the definitions, the function γ depends on r , so that the precise restriction on r implied by property (F) depends on the form of the utility function. We consider some standard examples below.

- Corollary 2 shows that fixed point \hat{x} is unique if $\gamma(x)$ is monotone in x . As $\gamma(x) < \gamma(0) = 1$, this condition implies $\gamma(\infty) = \inf_x \gamma(x)$. A sufficient condition for the monotonicity of γ is the property that the curvature of U , measured as $-xU''(x)/U'(x)$, be non-decreasing in x .

3.3. Examples

We set out examples of bequest functions and their properties with standard specifications for U . It is not possible to solve for the function $\phi(x)$ in closed form in Examples 1 and 2, though we can evaluate the restrictions implied by property (F). Example 3 has a quadratic utility. The bequest function is linear and can be solved explicitly in terms of the parameters of the system.

EXAMPLE 1 (Constant Relative Risk Aversion). Let $U(c) = c^\alpha/\alpha$ for $\alpha < 1$. Note that

$$\gamma(x) = \left[\frac{w + rx}{w + (1+r)x} \right]^{1-\alpha}$$

which is decreasing in x , and $\lim_{x \rightarrow \infty} \gamma(x) = (r/1+r)^{1-\alpha} = \inf_x \gamma(x)$.

The function ϕ has property **(F)** if and only if $1 > \theta(r) > (r/1+r)^{1-\alpha}$. The unique fixed point is $\hat{x} = (w(1 - \theta^{1/1-\alpha})) / (\theta^{1/1-\alpha}(1+r) - r)$.

EXAMPLE 2 (Constant Absolute Risk Aversion). Let $U(c) = 1 - \exp(-c)$. We have

$$\gamma(x) = \exp(-x),$$

which is decreasing in x , and $\lim_{x \rightarrow \infty} \gamma(x) = \inf_x \gamma(x) = 0$.

The function ϕ has property **(F)** if and only if $1 > \theta(r) > 0$. The unique fixed point is $\hat{x} = -\ln(\theta)$.

EXAMPLE 3 (Quadratic Utility Functions). Let

$$U(c) = -\frac{(C_* - c)^2}{2} \quad \text{for } c \in (0, C_*).$$

This utility function has a satiation point C_* ; we will need to establish that solutions have the property that $c_t \in (0, C_*)$. A first condition is

$$\text{(Q1)} \quad C_* > w.$$

The Euler equation for an interior optimum is

$$\beta\pi_{11}(1+r)x_{t+2} - (1 + \beta(1+r))x_{t+1} + (1+r)x_t = (C_* - w)(1 - \beta(1+r))$$

which is just $G(x, \phi(x)) = 0$ with $x_{t+1} = \phi(x_t)$. The two characteristic roots are the zeros of the polynomial

$$P(\lambda) = -\beta\pi_{11}(1+r)\lambda^2 + (1 + \beta(1+r)^2)\lambda - (1+r).$$

The saddlepoint property holds if and only if the lower root is real and less than 1, which is satisfied if and only if

$$\text{(Q2)} \quad P(1) > 0 \Leftrightarrow (1+r) < \frac{1}{1 + \beta\pi_{11} - \beta(1+r)}.$$

The stable root is $\lambda = B - (B^2 - 1/\beta\pi_{11})^{1/2}$, where $B = (1 + \beta(1+r)^2) / (2\beta\pi_{11}(1+r))$. The unique fixed point is

$$\hat{x} = (C_* - w) \frac{\beta(1+r) - 1}{P(1)}$$

which is positive only if

$$\text{(Q3)} \quad \beta(1+r) > 1.$$

The bequest function is given by

$$\phi(x) = \lambda x + (1 - \lambda) \hat{x}.$$

The solutions are interior whenever $0 < c_{1t} < c_{0t} = w + (1 + r) x_t < C_*$. This holds for each $x_t \in [0, \hat{x}]$ whenever

$$(Q4) \quad \beta(1 + r) < \frac{1}{\pi_{11}}.$$

Note that (Q2), (Q3), (Q4) are equivalent to

$$1 < \beta(1 + r) < \frac{1}{\pi_{11}}$$

which is simply the condition that $0 < \theta(r) < 1$, the condition equivalent to (F), when $\gamma(x)$ decreases to 0.

These examples have the property that γ is monotonic, and the fixed point is unique. Note that \hat{x} is decreasing in θ , so that \hat{x} is finite whenever the lower bound on θ applies. It is known (e.g. Kurz, (*op. cit.*)) that the policy function of a dynamic optimization program with wealth effects can have multiple fixed points even with standard assumptions; the non-decreasing curvature condition holds in all examples above.

4. THE STATIONARY DISTRIBUTION OF WEALTH

A stationary distribution of wealth is one component of a stationary equilibrium. In Theorem 5, we characterize the stationary distribution of wealth, given w , r and the Markov process for $s(h, t)$.

Suppose Assumption 1 holds, and let $w > 0$ and $r_t = r > -1$ at each t . A stationary bequest function $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ exists. An altruist with inheritance x chooses bequest $\phi(x)$.

We first define a candidate probability distribution, which is discrete and show in Theorem 5 that this distribution is in fact the unique stationary distribution of wealth.

DEFINITION 3 (Distribution F_ϕ). Let $\phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a stationary bequest function. Let $\mathcal{X} = \{x_j \geq 0; j = 0, 1, 2, \dots\}$, where

$$x_0 \equiv 0 \quad x_j \equiv \phi(x_{j-1}) \quad j \geq 1.$$

F_ϕ is a discrete probability distribution such that $\mathcal{X} = \text{supp } F_\phi$ and

$$f_0 \equiv \text{Prob}(x = x_0) = 1 - \pi;$$

$$f_j \equiv \text{Prob}(x = x_j) = (1 - \pi)(1 - \pi_{00}) \pi_{11}^{j-1} \quad \text{for } j = 1, 2, \dots$$

The distribution F_ϕ is fully characterized by the array $\{(x_j, f_j): j = 0, 1, 2, \dots\}$. Note that $f_j > 0$ whenever $0 < \pi_{ss} < 1$ for $s \in \{0, 1\}$, and that $\sum_j f_j = (1 - \pi)((1 - \pi_{11}) + (1 - \pi_{00}))/((1 - \pi_{11})) = 1$. Further, that $\phi(0) = 0 \Rightarrow x_j = 0$ for all $j = 0, 1, 2, \dots$, and that F_ϕ is non-degenerate whenever $\phi(0) > 0$.

The frequencies f_j are declining in j , that is, $f_{j+1} < f_j$. This corresponds to a "declining frequency" on x , implying that larger wealth holdings are less frequent, if $x_{j+1} > x_j$. We show that this indeed the case, that is, that the support of ϕ is restricted to x such that $\phi(x) \geq x$. We first need a Lemma to establish whether x_j is bounded, which is related to property (F).

LEMMA 4. *Suppose $\phi(0) > 0$. \mathcal{X} is bounded, with $x_j < \bar{x} < \infty$ if, and only if, $\phi(x)$ has property (F).*

Proof. Suppose property (F) holds. Then $x_1 = \phi(0) > 0 = x_0$ from Lemma 2. ϕ is strictly increasing by Theorem 2, so that $\hat{x} > 0 \Rightarrow \phi(\hat{x}) > x_1$. Let $\hat{x} = \phi(\hat{x})$. Then $x_0, x_1 < \hat{x}$. By induction, $x_{j+1} = \phi(x_j) < \hat{x}$ whenever $x_j < \hat{x}$ and $\phi(\hat{x} = \hat{x})$.

Suppose (F) does not hold. Then, $\phi(x) > x$, implying $\lim_{x \rightarrow \infty} \phi(x) = \infty$, as well as $x_j > x_{j-1}$ and $\lim_j x_j = \lim_j \phi^j(0) = \infty$. ■

THEOREM 4 (Declining Frequencies). *The distribution F_ϕ the property that $f_{j+1} < f_j$ and $x_{j+1} > x_j$ for $j \geq 0$, whenever $\beta(1+r) > 1$.*

Proof. The property that $f_j < f_{j+1}$ is evident from the definition of f_j , noting that $\pi_{ss} \in (0, 1)$. If $\beta(1+r) > 1$, we know that $\phi(0) > 0$ from Theorem 2.

If ϕ does not have property (F), $\phi(x) > x$ for all $x \geq 0$, so $x_{j+1} > x_j$. Suppose property (F) does hold. Note, from the Proof of Lemma 4, that $x_j < \hat{x}$ whenever $\hat{x} = \phi(\hat{x})$. Let $\bar{x} = \min_x \{\phi(x) = x\}$. The function $d(x) = \phi(x) - x$ is continuous, with $d(0) > 0$ and $d(\bar{x}) = 0$. It follows that $d(x) > 0$ for each $x < \bar{x}$, which proves the result. ■

We can write the distribution function $F_\phi(x)$ as a step function:

$$F_\phi(0) = f_0 = 1 - \pi;$$

$$F_\phi(x) = \sum_{j=0}^k f_j = 1 - \pi \pi_{11}^k \quad \text{for } x_k \leq x < x_{k+1};$$

for $k = 0, 1, 2, \dots$

THEOREM 5 (The Distribution of Wealth). *The discrete probability distribution F_ϕ is a stationary distribution of wealth. It is the unique stationary distribution. The stationary distribution is non-degenerate if, and only if, $(1+r) > 1/\beta$. Otherwise, F_ϕ is degenerate at $x=0$. The distribution of wealth converges to F_ϕ irrespective of the initial distribution F_0 .*

We give a constructive proof.

For each family h , let $n(h, t) = \min\{i: 0 \leq i < t \text{ such that } s(h, t-1-i) = 0\}$, and set $n(h, t) = t$ if $s(h, t-1-i) = 1$ for $0 \leq i < t$. Family h is of type n at t if $n(h, t) = n$. The index n measures the time elapsed since the last time the event $s(h, t) = 0$ occurred, equivalently, the number of successive altruist generations.

LEMMA 5. *Let $q_t(n) = \text{Prob}(n(h, t) = n)$ for $n = 0, 1, \dots, t$. Then*

$$q_t(n) = f_n$$

for each $n < t$, and

$$q_t(t) = \pi \pi_{11}^{t-1}$$

for each $t > 1$.

Proof. Let $t > 1$, and consider $0 < n < t$. The event $n(h, t) = n$ is equivalent to the event $I_n \equiv [s(h, t-n) = 0, s(h, t-n+i) = 1 \text{ for } 1 \leq i \leq n]$. Event I_n occurs with probability f_n , irrespective of t , whenever $0 < n < t$. The probability of the event $I_t \equiv [s(h, t-i) = 1 \text{ for each } i = 0, 1, \dots, t-1]$ is $\pi \pi_{11}^{t-1}$. ■

The next lemma establishes that bequests depend only on the index $n(h, t)$ as long as $n < t$.

LEMMA 6. *For each (n, t) such that $t > 1$ and $0 \leq n < t$, $n(h, t) = n \Rightarrow x_t^h = x_n$.*

Proof. Let $t > 1$. If $n(h, t) = 0$, the bequest is $x_t^h = 0 = x_0$. The recursive relation $x_t^h = \phi(x_{t-1}^h)$ implies $x_t^h = x_j$ whenever $x_{t-1}^h = x_{j-1}$ and $s(h, t-1) = 1$. ■

We show next that the distribution of bequests converges to F_ϕ pointwise on \mathbb{R}_+ .

LEMMA 7. *Let F_t be the distribution of bequests at t . For each $x \in \mathbb{R}_+$,*

$$\lim_{t \rightarrow \infty} |F_t(x) - F_\phi(x)| = 0$$

Proof. Let F_0 be the initial distribution of bequests received by the first generation. Define F_{0t} as the distribution of $\phi^t(x)$ induced by F_0 . From Lemma 5, proportion $q_t(t)$ of the population has $n(h, t) = t$. From Lemma 6, $x_t^h = x_n$ whenever $n < t$. It follows that

$$F_t(x) = F_\phi(x) + q_t(t) F_{0t}(x)$$

whenever $x \leq \phi^{t-1}(0)$, and

$$F_t(x) = (1 - q_t(t)) + q_t(t) F_{0t}(x)$$

if $x > \phi^{t-1}(0)$. The proof is completed by noting that $\lim_{t \rightarrow \infty} q_t(t) = 0$ and that $F_{0t}(x)$, $F_\phi(x)$ are bounded for each $x \in \mathbb{R}_+$. ■

We have shown that F_ϕ is the only candidate for a stationary distribution of bequests. To establish that it is a stationary distribution, suppose $F_t(\mathcal{X}) = 1$, with $f_{j,t} = \text{Prob}_t(x_j)$. From the definitions, $f_{0,t} = (1 - \pi)$ for each t ; and that $f_{1,t+1} = (1 - \pi_{00}) f_{0,t}$, and $f_{j,t+1} = \pi_{11} f_{j-1,t}$ for each if $j > 1$ and all t . F_ϕ is the only distribution on \mathcal{X} which satisfies these restrictions.

Finally, $\phi(0) > 0$ implies $x_1 \neq x_0$ so that F_ϕ puts positive probability on both as long as $0 < \pi_{ss} < 1$. If $\phi(0) = 0$, $x_j = 0$ for each j so that $F_\phi(0) = 1$. From Theorem 2, this equivalent to $\beta(1+r) \leq 1$. ■

4.1. Remarks

- The bequest function either has a fixed point at 0; failing this, it can possess property (F), which implies at least one positive fixed point; otherwise, it has no fixed point and $\phi(x) > x$ for all $x \geq 0$. These three cases have quite different implications for the wealth distribution F_ϕ .

- If $\phi(0) = 0$, which occurs if $r \leq (1 - \beta)/\beta$, the stationary distribution is necessarily degenerate at 0. Even if we start with a distribution F_0 , over time, the proportion of households with positive inheritance decays to 0.

- If ϕ has property (F), the support of F_ϕ is bounded, with $x_j < \bar{x}$ where \bar{x} is the smallest fixed point of ϕ .

- If $\phi(x) > x$ for each $x \geq 0$, the support of F_ϕ is unbounded: for every positive wealth level x , there is a small but positive proportion of the population whose wealth exceeds it.

- The Pareto distribution is known to approximate the distribution of wealth in different countries and at widely different times. (e.g. Persky [22]). The Pareto Distribution with parameter a has

$$f_j = Ax_j^{-a}$$

for some $a > 0$. In our distribution, $x_j = \phi^j(0)$, so that the Pareto law would hold exactly if bequest were proportional, i.e. $\phi(x) = c_0 x$. The Pareto property holds asymptotically if,

$$\lim_{j \rightarrow \infty} \left| \frac{x_{j+1}}{x_j} \right| = c_0 > 0.$$

We note that this property is satisfied by ϕ in our explicit example (Example 3) with quadratic utilities. Wold and Whittle [26] show that such a distribution is compatible with a Markov process for wealth, with division of estates at death.

• Åbel and Bernheim [2] establish that the proposition of Ricardian equivalence fails if individuals are imperfectly altruistic, because the response of bequest levels to tax-financed public borrowing does not fully offset the effects of such a policy. This continues to be true here. Consider a temporary fiscal policy indexed by τ , which consists of uniform lump-sum transfers in periods t and $t+1$ such that $t_t^h = \tau$ and $t_{t+1}^h = -(1+r)\tau$. This policy will affect consumption and wealth at t : families with $s(h, t) = 0$ increase their consumption by the full amount, and aggregate wealth X_t , must fall by the quantity $\pi\tau$. Such a policy represents a temporary transfer to non-altruistic agents. Similarly, a permanent policy of (say) a stationary level of public debt financed by taxes will alter consumption levels as well as the stationary distribution of wealth.

5. STATIONARY EQUILIBRIUM AND CONVERGENCE

We have defined a stationary equilibrium as a pair $K \geq 0$, $Y > 0$ and a distribution F such that F is the stationary distribution of wealth, and

$$K = \int x dF$$

is the stationary level of capital stock, with $Y = \rho K + w$. We know, from Theorem 5, that a stationary distribution F_ϕ exists. This yields a stationary equilibrium whenever

$$\int x dF_\phi < \infty.$$

We define a condition which ensures this.

DEFINITION 4 (Property **B**). The bequest function ϕ property **(B)** if

$$\sum_{j=1}^{\infty} \pi_{11}^j \phi^j(0) < \infty.$$

Remarks. • We can write

$$\kappa \equiv \int x dF_{\phi} = (1 - \pi_{00}) \frac{(1 - \pi)}{\pi_{11}} \sum_{j=1}^{\infty} \pi_{11}^j \phi^j(0)$$

and note that $\kappa < \infty$ if and only if ϕ has property **(B)**.

• If $\beta(1+r) \leq 1$, we have $\phi(0) = 0$ and F_{ϕ} degenerate at 0. It follows that $\kappa = 0$, and property **(B)** holds trivially.

• Similarly, **(F)** \Rightarrow **(B)**. From Lemma 4, $x_j < \bar{x} < \infty$ whenever property **(F)** holds, implying $\kappa < \bar{x} < \infty$.

• Note further that **(B)** can be true even when **(F)** fails. Consider, for example, $\phi(x) = a + bx$, with $a > 0$. Property **(F)** is equivalent to $b < 1$, whereas **(B)** is equivalent to $b < 1/\pi_{11}$.

• Condition **(B)** requires that x_j grow slower than $1/\pi_{11}^j$. A sufficient condition for this is

$$\pi_{11}(1+r) < 1.$$

This restriction follows from the observation $\phi(x) < w + (1+r)x$ from Corollary 1. This is compatible with the failure of **(B)**, which can occur if $\beta(1+r) > 1/(\pi_{11} + (1 - \pi_{11}) \inf_x \gamma(x))$.

To show that capital stocks K_t converge to the stationary value κ , we make a strong assumption that $\gamma(x)$ is decreasing in x . This implies, of course, $\phi(x)$ has at most one fixed point: it may have none if property **(F)** fails.

THEOREM 6 (Convergence). Suppose $\gamma(x)$ is decreasing in x , and let F_0 be the distribution of wealth at $t=0$, where $F_0(x^*) = 1$ for some $x^* < \infty$. Then, $\lim_{t \rightarrow \infty} K_t = \kappa \equiv \int x dF_{\phi}$ whenever ϕ has property **(B)**.

Proof. Property **(B)** implies $\kappa < \infty$. Recall from the proof of Lemma 7 that

$$F_t(x) = F_{\phi}(x) + q_t(t) F_{0t}(x)$$

if $x \leq \phi'^{-1}(0)$ and

$$F_t(x) = (1 - q_t(t)) + q_t(t) F_{0t}$$

otherwise. Let $Z_t = \int x dF_{0t}$, and $X_t = \sum_{j=0}^t f_j x_j$. Then,

$$K_t = \int x dF_t = q_t(t) Z_t + X_t.$$

Note that $q_t(t) \rightarrow 0$ and $X_t \rightarrow \kappa$ as $t \rightarrow \infty$. It remains to show that $q_t(t) Z_t \rightarrow 0$ as $t \rightarrow \infty$.

If ϕ has property **(F)**, we know that it has a unique fixed point \hat{x} , and $Z_t \leq \phi^{t-1}(x^*) < \max[\hat{x}, x^*]$. Clearly, $q_t(t) Z_t \rightarrow 0$.

If ϕ does not have property **(F)**, we have $\phi'(0) \rightarrow \infty$, and $\phi(x) > x$. For some T , $\phi^{t+T}(0) > x^*$ if $t \geq 0$. Property **(B)** implies $q_t(t) Z_t \leq \pi_{11}^{t+T} \phi^{t+T}(0) \rightarrow 0$. ■

THEOREM 7 (Asymptotic Growth). *Let F_0 be the distribution of wealth at $t=0$, where $K_0 = \int x dF_0 < \infty$. Then, $\lim_{t \rightarrow \infty} K_t = \infty$ whenever ϕ does not have property **(B)**.*

Proof. Let X_t, Z_t be as defined in Theorem 6, and note that $K_t = X_t + q_t(t) Z_t$, that $X_t \rightarrow \infty$ as $t \rightarrow \infty$ and that $q_t(t) Z_t \geq 0$ at each t . ■

Remarks. • Theorems 6 and 7 establish that the sequence of capital stocks converge to κ , irrespective of whether it is finite.

- The monotonicity condition on γ , equivalent to the non-decreasing curvature of U , is stronger than we require. It allows for a relatively simple proof.

- If ϕ does not have property **(B)**, the quantity of output and capital must diverge with time. It is possible to show that for some initial distributions, F_0 , this path is monotone, implying perpetual growth.

- Condition **(B)** fails only if r is sufficiently large: it is necessary that $(1+r) > 1/\pi_{11} \gg \beta$. It is known that the optimal path of capital accumulation in a discounted dynamic program can diverge if $\lim_{K \rightarrow \infty} Y'(K)$ is large enough (e.g. Jones and Manuelli [14]) as here.

- As an organizing device, it is useful to think of three regions for r , which are intervals $I_0 \equiv (0, R_0]$, $I_B \equiv (R_0, R_1]$ and $I_\infty \equiv (R_1, \infty)$: define $R_0 = (1-\beta)/\beta$ and R_1 such that condition **(B)** holds whenever $r \leq R_1$. We know, from Theorems 2, 5, and 6, that $r \in I_0$ implies that capital stocks decline to zero asymptotically, as $\kappa = 0$. From Theorem 6, $r \in I_B$ implies that capital stocks converge to the stationary value κ over time, whereas $r \in I_\infty$ implies asymptotic growth by Theorem 7.

- The usual condition for stationary is $r \in I_0$ (e.g. Jones and Manuelli (*op. cit.*)), which is equivalent to $K_t \rightarrow 0$. The randomness of $s(h, t)$ is crucial for the existence of the intermediate interval I_B , where stationary capital stocks are positive and finite.

6. RELATED LITERATURE

There are two distinct strands of literature which this paper relates to. The first corresponds to theories explaining the distribution of wealth, and of human capital. The second strand contains attempts to reconcile the quite different properties of overlapping generations models and models of infinitely lived households. We consider these in turn, and in relation to our model.

6.1. *The Distribution of Wealth and Income*

Theories explaining the distribution of wealth are based either on life cycle saving of individuals facing uninsured income uncertainty (e.g. Huggett [13], among others), or on the heterogeneity of inheritances attributed to differences in the history of income or mortality in the family (e.g. Becker [7], Loury [18]), Abel [1], Eckstein, Eichenbaum and Peled [11], and Galor and Zeira [12]). In a sequence of papers, Laitner [16, 17], emphasizes the need to combine the two characteristics for a realistic theory of levels, and evolution, of wealth holdings.

The approach closest to ours is that of accidental bequests, as set out in Abel (*op. cit.*) and Eckstein *et al.* (*op. cit.*). Indeed, the positive aspects of these theories are very similar, especially in their predictions about the rise and fall of family fortunes. This similarity is not true for normative aspects, for reasons which we explain shortly.

In the following, we oversimplify the accidental bequest argument in order to compare the predictions to ours. Imagine that $\pi = 0$. As before, total wealth of individual (h, t) is $x_{t-1}^h(1+r) + w$, which equals intended consumptions. However, individuals may die either before, or after, consuming this amount. In the former case, their heir inherits this wealth, so that

$$x_t^h \in \{(1+r)x_{t-1}^h + w, 0\}.$$

This is essentially the same as the evolution rule of our model, with a linear bequest function. Supplemented by a stochastic process on the event of survival, i.e., that $x_t^h = 0$, yields a stationary distribution in much the same way as ours. The assumed linearity of ϕ allows a full characterization of the stationary distribution, F_ϕ . Extensions of this model (to T periods of life and changing probabilities of survival) lead to more states per family, and to a more complicated bequest function. The basic mathematical argument, which relies on the process x_t^h renewing itself with positive probability remains unchanged.

A similar argument can be made with respect to income uncertainty. We have not seen this made elsewhere, and briefly indicate the argument.¹

¹ We would like to thank Jim Mirrlees for suggesting this interpretation.

Imagine that individuals (h, t) face shocks with respect to effective labor endowment, so that $\ell_t^h \in \{\ell_0, \ell_1\}$, $\ell_0 < \ell_1$. Assume now that $\pi = 1$, so that all individuals are similarly altruistic, and that shocks to labor endowments follow a stationary Markov process. It is now possible to work out stationary bequest functions $\phi_0(x)$, $\phi_1(x)$ such that individuals with labor endowment ℓ_i and inheritance $(1+r)x$ leave $\phi_i(x)$ to their children. To apply the argument exactly as here, we need to assume that the parameters are such that $\phi_0(0) = 0 < \phi_1(0)$. A parallel assumption is made by Loury (*op. cit.*).

The stationary distribution of wealth falls in our class of F_ϕ distributions, for appropriate ϕ , in both cases. In our model, the heterogeneity across families is with respect to preferences; the corresponding bequest decisions are optimal, and the stationary distribution corresponds to a Pareto efficient allocation. In the case of accidental bequests, this is no longer true. Indeed, there is room for an annuities market. Actuarially fair annuities would simultaneously improve welfare and make the distribution of bequests degenerate at 0. Similarly, wealth distributions due to income uncertainty are sustained by incomplete markets, which do not allow individuals to insure against fluctuations in labor endowments. With incomplete markets, it is possible to evaluate Pareto-improving inheritance taxes. In our framework, this is ruled out, and inheritance taxes have to be justified by appeal to a preference for equality on behalf of the social planner.

This discussion strikes a cautionary note about the possibility of characterizing the observational implications of efficiency, at least in the case of wealth or income distributions. The distribution of wealth does not provide enough information to judge the desirability of redistributive taxation.

Laitner [17] evaluates the role of bequest and life-cycle savings in explaining the distribution of wealth. The model there is much richer than ours. The major result there is similar to the following in ours. Suppose $\pi = 1$. Then, the steady state distribution of wealth is necessarily degenerate at \hat{x} , which is positive only if $\phi(0) > 0$. An operative bequest motive is not compatible with a non-trivial distribution in the long run. This insight suggests that a reasonable theory of wealth distributions should assume additional sorts of heterogeneity. We rely on differences in preference; Laitner [16] uses steady-state earnings differences across families, where $w_t^h = w^h$ for each t .

In our analysis, the bequest decision consists of the choice of physical capital which then earns a certain rate of return. Loury [18] studies the implications of a bequest of human capital. Parents incur expenditure on education, which earns a random return based on children's abilities. This last varies across families and over time, which is the driving force in deriving the stationary distribution. In a similar framework, Galor and Zeira [12] assume that the technology of human capital bequests is deterministic but non-convex. In our terminology, this results in a ϕ function

which has multiple fixed points; the initial distribution of human capital determines the transition path, and thus selects a long-run equilibrium. It would be interesting to combine this type of hypotheses, with ours, to evaluate the possibilities of long-run growth and equality in dynamic economies.

6.2. *Dynamic Economies and Altruistic Links*

Dynamic macroeconomic analysis is usually conducted in one of two quite different frameworks—that of the infinitely-lived representative agent, and that of overlapping generations. The analysis often reaches very different conclusions in the two models, with respect to the determinacy, and efficiency of equilibrium paths, as well as with respect to the relevance or desirability of economic policies. For example, the role of national debt, and the proposition of Ricardian equivalence, are very distinct in the analyses of Barro [6] and Diamond [10].

The paradigm of infinitely-lived households is often justified as an approximation to situations where finitely-lived individuals have altruistic preferences towards their descendants. To establish results on efficiency and determinacy, and policy-neutrality, it is important that every household face a single budget constraint, over time and states of nature. Bequests provide this link, provided, of course, that desired bequests are positive in every generation. Otherwise, neither efficiency nor neutrality can be asserted (Weil [25]). The degree of altruism, β , is a crucial parameter in determining the extent to which the bequest motive is operative. In our model, it simultaneously indexes the degree of inequality in the distribution of wealth. The stationary distribution is degenerate if β is sufficiently small. For β large enough, the economy experiences perpetually increasing inequality.

To the best of our knowledge, there are two other ways in which one achieve a model intermediate between overlapping generations and infinitely lived ones. The first relies on preference heterogeneity across families (e.g. Becker [7], Aiyagari [4, 5], or Muller and Woodford [21]). The second, on uncertain, but finite lifetimes (Yaari [27], Blanchard [9], and Abel (*op. cit.*)); we have already described the implication of the lifetime uncertainty assumption for wealth distributions. The Blanchard–Yaari model differs in technical details, such as continuous rather than discrete time; the economic implications are distinct because of their assumption that $\pi = 0$, that is, there are never any bequest motives.

To evaluate the role of preference heterogeneity, imagine a family of infinitely lived households, which differ in their taste for altruism, as described by $\beta^h \in [0, 1]$. Unlike our model, this assumes that tastes are inherited, i.e. $\pi_{ii} = 1$. A special case of this is the assumption that $\beta^h \in \{0, \beta\}$, as in Muller and Woodford (*op. cit.*) as well as Michel and Pestieau (1994). It is known since Ramsey [23] at least that in such a world, all the wealth of the economy is eventually owned by the most patient household, which has obvious

implications for the analysis of stationary equilibria. The property no longer holds in the presence of non-negative bequest constraints, or other forms of borrowing constraints. Becker (*op. cit.*) established the existence of non-degenerate stationary distribution in that framework. Once again, the characteristics of the distribution are easily described in our terminology. We assume that β^h differs across h but not t . It is now possible to define bequest functions $\phi^h(x)$, from the policy function of the discounted dynamic program with discount factor β^h , given all other parameters. Let $\hat{x}(\beta^h)$ be a fixed point of ϕ^h . The stationary distribution of wealth must live on $\{\hat{x}(\beta^h); \beta^h \in [0, 1]\}$. The stationary distribution of wealth is thus a fairly straightforward transformation of the fixed distribution on preferences.

The work of Aiyagari [4, 5] offers a more intriguing possibility. In an economy with two types of preferences, $0 < \beta_L < \beta_H < 1$, he shows that there can be multiple stationary equilibria. For certain values of the parameters, there exists an equilibrium where both types leave positive bequests, and another equilibrium where neither does. This multiplicity requires that interest rates be determined each period, rather than parametric. Roughly speaking, the phenomenon occurs if the period utility function $U(\cdot)$ has deep enough curvature to entail non-monotone savings responses to interest rates. Extension of our framework to the neoclassical growth model may well display this as a robust phenomenon, with quite strong implications about the evaluation of Ricardian policies and of inheritance taxes.

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